

Transition from lognormal to χ^2 -superstatistics for financial time series

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Share price returns on different time scales can be well modeled by a superstatistical dynamics. Here we provide an investigation which type of superstatistics is most suitable to properly describe share price dynamics on various time scales. It is shown that while χ^2 -superstatistics works well on a time scale of days, on a much smaller time scale of minutes the price changes are better described by lognormal superstatistics. The system dynamics thus exhibits a transition from lognormal to χ^2 superstatistics as a function of time scale. We discuss a more general model interpolating between both statistics which fits the observed data very well. We also present results on correlation functions of the extracted superstatistical volatility parameter, which exhibits exponential decay for returns on large time scales, whereas for returns on small time scales there are long-range correlations and power-law decay.

I. INTRODUCTION

Many well established concepts in mathematical finance (such as the Black-Scholes model) are based on the assumption that an index or a stock price follows a geometric Brownian motion, and as consequence the log returns of these processes are Gaussian distributed. But nowadays it is well known that the log returns of realistic stock prices are typically non-Gaussian with fat tails [1]–[22]. Such behaviour can be well captured by superstatistical models [2]–[15]. The basic idea of this method borrowed from nonequilibrium statistical mechanics is to regard the time series as a superposition of local Gaussian processes weighted with a process of a slowly changing variance parameter, often called β . This approach has been applied to many areas of complex systems research, including turbulence, high energy scattering processes, heterogenous nonequilibrium systems, and econophysics (see e.g. [11] for a short review). In finance early applications of the superstatistics concept were worked out by Duarte Queiros et al. [6, 7] and Ausloos et al. [5]. Van der Straeten and Beck [3] analysed daily closing prices of the Dow Jones Industrial Average index (DJI) and the SP 500 index. They verified that both log-normal superstatistics and χ^2 superstatistics result in good approximations. Biro and Rosenfeld [4] also studied the data sets of the Dow Jones index and verified that the distribution of log returns is well fitted by a Tsallis distribution. Katz and Li Tian [1] showed that the probability distributions of daily leverage returns of 520 North American industrial companies during the 2006-2012 financial crisis comply with the q -Gaussian distribution which can be generated by χ^2 superstatistics. They also verified in [2] that the Tsallis entropic parameter q obtained by direct fitting to q -Gaussians coincides with the q obtained from the shape parameters of the χ^2 distribution fitted to the histogram of the volatility of the returns. Gerig, Vicente and Fuentes [8] consider a similar model that indicates that the volatility of intra day returns is well described by the χ^2 distribution, see also [9] for related work in this direction.

In this paper, we will carefully analyse for various data

sets of historical share prices which type of superstatistics is best suited to model the dynamics. While Tsallis statistics ($= q$ -statistics) is known to be equivalent to χ^2 superstatistics [13, 22], there are other types of superstatistics, such as lognormal superstatistics and inverse χ^2 superstatistics [10], which are known to be different from q -statistics (though all these different statistics generate similar distributions if the variance of the fluctuations in β is small [13]). We show that in our analysis χ^2 -superstatistics appears best suitable to describe the daily price changes, whereas on much smaller time scales of minutes lognormal superstatistics seems preferable. We analyse the relevant time scale of the changes in the superstatistical parameter β and present results for the decay of correlations in β . For small return time scales, correlation functions exhibit power law decay and there are long memory effects. In the final section, we develop a synthetic stochastic model that fits the data well. This is kind of a hybrid model interpolating between lognormal and χ^2 -superstatistics.

This paper is organized as follows. In section II we look at share price returns on large (daily) time scales. In section III we do a similar analysis on small (minute) time scales. In section IV we investigate correlations of the superstatistical volatility parameter on both time scales. In section V the hybrid model is introduced. Our final concluding remarks are given in section VI.

II. SUPERSTATISTICS OF LOG-RETURNS OF SHARE PRICES ON A LARGE TIME SCALE

Non-equilibrium system dynamics can often be regarded as as superposition of a local equilibrium dynamics and a slowly fluctuating process of some variance variable β [13]. These types of ‘superstatistical’ nonequilibrium models are also useful for financial time series [6, 7]. In this article, the empirical data we use as an example is the historical stock prices of Alcoa Inc(AA), which is an American company that engages in the production and management of primary aluminium, fabricated aluminium and alumina. We have looked at shares of many other companies as well (see Tab. 1 in section IV), with

similar results. Our data set covers the period January 1998 to May 2013. We study the log return R_i denoted by

$$R_i = \log \left(\frac{S_{i+1}}{S_i} \right) \quad (1)$$

where $i = 0, 1, 2, \dots, N$; S_i and S_{i+1} are two successive daily closing prices. We consider the normalised log returns

$$u_i = \frac{R_i - \langle R \rangle}{\sqrt{\langle R^2 \rangle - \langle R \rangle^2}} \quad (2)$$

which have been rescaled to have variance 1. The symbol $\langle \dots \rangle$ denotes the long-time average.

From the simplest superstatistics model point of view, the entire time series of stock prices can be divided into n smaller time slices T . We call T optimal window size. Within each T , the financial volatility β is temporarily constant and the log return of the stock price is Gaussian distributed. β has some probability distribution $f(\beta)$ to take a particular value in a given slice. The conditional probability $p(u|\beta)$ is

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp \left(-\frac{1}{2}\beta u^2 \right) \quad (3)$$

and the marginal probability distribution of u for long time observation is the average over local Gaussians weighted with the probability density $f(\beta)$

$$p(u) = \int p(u|\beta) f(\beta) d\beta. \quad (4)$$

The integration over β yields non-Gaussian behaviour with fat tails.

We now describe our technique to obtain the optimal window size T for a given time series. Firstly we split the time series into

$$n = \lfloor \frac{N}{\Delta t} \rfloor \quad (5)$$

equal intervals, where $\lfloor \cdot \rfloor$ denotes the floor function and Δt is the dimensionless window size, i.e. the number of data points in a given window. N is the total number of data points of the entire time series. Generally the kurtosis of a random variable u is defined as

$$\kappa = \frac{\langle u^4 \rangle}{\langle u^2 \rangle^2} \quad (6)$$

and it is equal to 3 for a Gaussian distribution of arbitrary variance. For a given window size Δt , the kurtosis in the j th window is given by

$$\kappa_{\Delta t}(j) = \frac{\frac{1}{\Delta t} \sum_{i=(j-1)\Delta t+1}^{j\Delta t} u_i^4}{\left(\frac{1}{\Delta t} \sum_{i=(j-1)\Delta t+1}^{j\Delta t} u_i^2 \right)^2}, \quad (7)$$

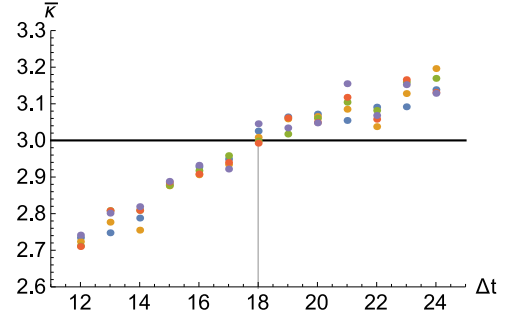


FIG. 1: Determination of the optimal window size for the Alcoa share price data. The intersection with the line kurtosis $\bar{\kappa} = 3$ yields $T = 18 \pm 0.5$. The various values of $\bar{\kappa}$ for a given Δt (indicated by different colors in the online version) are obtained for different translational shifts of the sliding windows. The scattering of the data can be used to estimate the standard deviation as $\delta\bar{\kappa} \sim 0.03$.

where $j = 1, 2, \dots, n$. When we have all the values of kurtosis for all windows, we can calculate an average kurtosis of the n windows as

$$\bar{\kappa}_{\Delta t} = \frac{1}{n} \sum_{j=1}^n \kappa_{\Delta t}(j). \quad (8)$$

The aim is to achieve an optimum window size such that for a given data set the distribution in each window is as close as possible to a Gaussian, but with varying variance. For this purpose the optimal window size T should satisfy the condition

$$\bar{\kappa}_{\Delta t} = 3. \quad (9)$$

Fig. 1 shows how the average kurtosis changes with the window size. We obtain from condition (9) the optimal window size 18 ± 0.5 for this example. The result makes financial sense. 18 trading days correspond to a time scale of about 3-4 weeks. It is a typical time scale where market volatility changes, due to events such as changes in the confidence in the future economic development, anticipated interest changes, and so on. See also [23] for related work.

With the given optimal window size, we can now calculate the local volatility parameter β in each time interval as

$$\beta_k = \frac{1}{\frac{1}{T-1} \sum_{i=(k-1)T+1}^{kT} (u_i - \bar{u}_i)^2} \quad (10)$$

where $k = 1, 2, \dots, n$. Note that the variance of u in each window is β^{-1} . One can then plot a histogram of the β_k and fit it with some suitable model distribution.

Here we will consider three distributions to be compared with our experimental distribution of β , which were previously advocated in [10]. The first one is the χ^2 -distribution for which $f(\beta)$ is given by

$$f_1(\beta) = \frac{1}{\Gamma\left(\frac{d_1}{2}\right)} \left(\frac{d_1}{2\beta_0}\right)^{d_1/2} \beta^{d_1/2-1} e^{-d_1\beta/2\beta_0}. \quad (11)$$

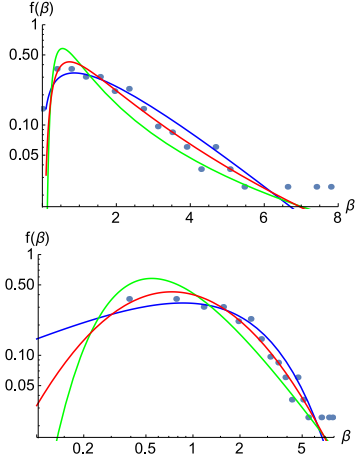


FIG. 2: Best possible fits that can be achieved for the distribution of the volatility β of Alcoa shares (plotted by dots), in a log-linear (top) and double logarithmic plot (bottom). Blue: χ^2 distribution $f_1(\beta)$ with $d_1 = 1.51, \beta_0 = 2.19$, Green: inverse χ^2 distribution $f_2(\beta)$ with $d_2 = 0.45, \beta_0 = 2.19$, Red: lognormal distribution $f_3(\beta)$ with $s = 0.87, \mu = 0.45$.

The second one is the inverse χ^2 -distribution where

$$f_2(\beta) = \frac{\beta_0}{\Gamma(\frac{d_2}{2})} \left(\frac{d_2 \beta_0}{2} \right)^{d_2/2} \beta^{-d_2/2-2} e^{-d_2 \beta_0 / 2\beta}. \quad (12)$$

The third distribution that will be tested is the lognormal distribution for which the probability density function is given by

$$f_3(\beta) = \frac{1}{\sqrt{2\pi}s\beta} \exp\left(-\frac{(\ln \beta - \mu)^2}{2s^2}\right) \quad (13)$$

where

$$\mu = \ln \beta_0 - \frac{s^2}{2}. \quad (14)$$

The β_0 in Eq. (14), (11), (12) is the mean value of β , given by

$$\beta_0 = \langle \beta \rangle = \frac{1}{n} \sum_{k=1}^n \beta_k, \quad (15)$$

and d_1, d_2, s are parameters. Lognormal superstatistics often occurs for complex systems described by a cascading dynamics [12], whereas χ^2 and inverse χ^2 superstatistics are more common for additive degrees of freedom contributing to a fluctuating temperature or inverse temperature [10].

We have fitted our experimental histograms $f(\beta)$ with the above distributions. Given β_0 , we vary d_1, d_2 and s of Eq. (11), (12), (13) in order to obtain the optimum fit to our observed $f(\beta)$. It can be seen in Fig. 2 that lognormal, χ^2 - and inverse χ^2 superstatistics all yield a more or less decent fit, though inverse χ^2 -superstatistics seems less favorable.

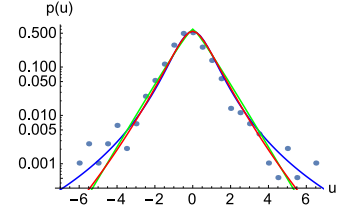


FIG. 3: Comparison of the histogram of u (plotted by dots) with the 3 types of superstatistics, integrated with the same parameters as in Fig. 2. Blue: χ^2 Superstatistics $p_1(u)$, Green: inverse χ^2 Superstatistics $p_2(u)$, Red: lognormal Superstatistics $p_3(u)$.

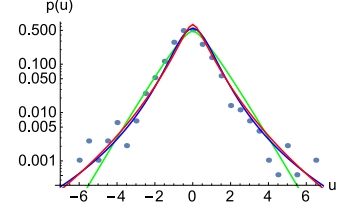


FIG. 4: Amended Superstatistics Blue: χ^2 Superstatistics $p_1(u)$ with $d_1 = 1.51, \beta_0 = 2.19$, Green: inverse χ^2 Superstatistics $p_2(u)$ with $d_2 = 1.2, \beta_0 = 2.19$, Red: lognormal Superstatistics $p_3(u)$ with $s = 1.2, \mu = 0.65$.

Still for consistency we also need to check the validity of Eq. (4). We thus also compare the original histogram of returns u with the following integrals where the parameters take the same values as in Fig. 2:

$$p_i(u) = \int \sqrt{\frac{\beta}{2\pi}} \exp\left(-\frac{1}{2}\beta u^2\right) f_i(\beta) d\beta \quad i = 1, 2, 3 \quad (16)$$

As shown in Fig. 3, for the integrated densities χ^2 superstatistics seems to fit better to the probability density of u compared with lognormal superstatistics and inverse χ^2 superstatistics.

Thus, if *independent* variation of the volatility parameter in each interval is assumed, then the data clearly point to χ^2 superstatistics, equivalent to Tsallis statistics [22]. On the other hand, independence of β_k may not always be a good approximation. There can be strong correlations of the volatility parameter β_k , and variations of the time scales where it is approximately constant. In that case more complicated dynamics arise, and one could then possibly get a better fit for the integrated distributions $p(u)$ if other effective parameters are used. For this reason, we also allowed the fitting parameters for $p_1(u), p_2(u), p_3(u)$ to take on other possible values. The result of this ‘amended superstatistics’ is shown in Fig. 4.

After the adjustment, we find in Fig. 4 that in fact *all three* superstatistics can describe $p(u)$ quite well. To distinguish between them, one would need much more data so that the tail behaviour would be clearer. In practice, more data are available if one considers price changes on much smaller time scales than days. This will be done in

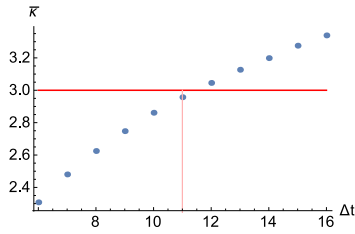


FIG. 5: Determination of the optimum window size for the 1-minute data of Alcoa. The intersection with the line *kurtosis* = 3 yields $T \approx 11$.

the next section.

III. SHORT TIME SCALES

Let us extend our analysis to returns on much smaller time scales. A change of statistics as a function of the time scale considered is a common phenomenon for many complex systems, see e.g. [25, 26] for work in this direction. Hence it is interesting to also consider return data on much smaller time scales (say, minutes), and see what is similar and what is different as compared to the analysis of the previous section. Let s_i be the stock price for every recorded minute, in our example chosen as that of Alcoa Inc(AA). The total number of data points is about 1.5 million. We look at the returns

$$r_i = \log \left(\frac{s_{i+\tau}}{s_i} \right) \quad (17)$$

where τ is an integer in units of minutes. The log returns are again normalized to variance 1:

$$u_i = \frac{r_i - \langle r \rangle}{\sqrt{\langle r^2 \rangle - \langle r \rangle^2}} \quad (18)$$

There is one small technical problem for these types of data, as the returns are not given overnight but only during normal working hours. This can lead to big overnight jumps and affect the analysis. For this reason, if $s_{i+\tau}$ and s_i are from two successive trading days, we removed the corresponding $\log \left(\frac{s_{i+\tau}}{s_i} \right)$. $\tau = 1$ means the log return is extracted every minute. Again we determined the optimal window size, using the same technique as in the previous section. We obtain $T \approx 11$ (see Fig. 5).

Again this time scale of about 11 minutes makes sense. It is a typical time scale on which new relevant information becomes available to the traders, leading to changes in the small-scale volatility. It also coincides with typical time scales on which observed correlations in short-term returns start to decay [24]. Our results of fitting the three types of superstatistics are shown in Fig. 6-8.

As can be seen in Fig. 6, the lognormal distribution is by far best fit of $f(\beta)$ if the time scale is 1 minute.

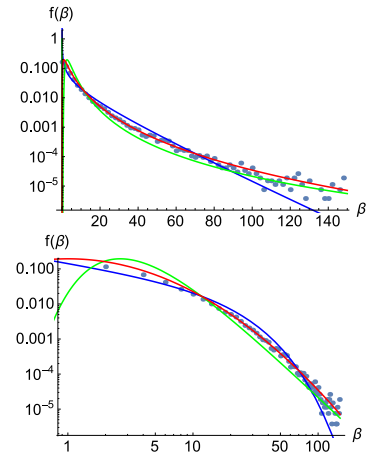


FIG. 6: Best fits that can be achieved for the distribution of the short-scale volatility parameter β (time scale of returns: 1 minute). Blue: χ^2 distribution $f_1(\beta)$ with $d_1 = 0.13, \beta_0 = 6.33$, Green: inverse χ^2 distribution $f_2(\beta)$ with $d_2 = 2.83, \beta_0 = 6.33$, Red: lognormal distribution $f_3(\beta)$ with $s = 1.11, \mu = 1.23$, top: log-linear plot, bottom: double logarithmic plot.

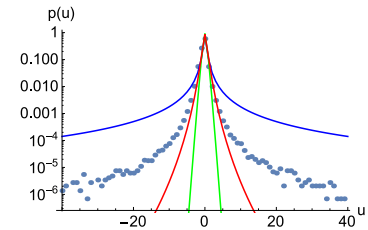


FIG. 7: Comparison of histogram of u (plotted by dots) with the integrated superstatistics distributions, using the same parameters as in Fig. 6. Blue: χ^2 Superstatistics $p_1(u)$, Green: inverse χ^2 Superstatistics $p_2(u)$, Red: lognormal Superstatistics $p_3(u)$. None of the curves is a good fit, indicating the presence of strong correlations for the volatility parameter β_k .

Fig. 7 shows a clear difference as compared to the daily data in Fig. 3: The integrated formula now does *not* give good fits to $p(u)$. The reason is that the β_k on a time scale of minutes are not anymore statistically independent, hence random sampling of Gaussians with different variance is not appropriate anymore.

After the free adjustment in the parameters of $p_1(u), p_2(u), p_3(u)$, again both χ^2 and lognormal superstatistics can provide good fits of $p(u)$. See Fig. 8.

If one does not allow for parameter amendments, then we can conclude that there is a transition from χ^2 to lognormal superstatistics when the time scale changes from 1 day to 1 minute. Also, a more general conclusion seems to be that the assumption of a sequence of independent volatility parameters β_k is not valid, as we are getting in general differences between the optimum fit of $f(\beta)$ and the corresponding fit of $p(u)$ written as an integral over Gaussians with the same corresponding parameters.

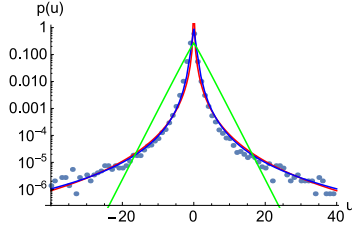


FIG. 8: Amended Superstatistics Blue: χ^2 Superstatistics $p_1(u)$ with $d_1 = 0.36, \beta_0 = 6.33$, Red: lognormal Superstatistics $p_3(u)$ with $s = 2.7, \mu = 3.9$, Green: inverse χ^2 -superstatistics ($d_2 = 0.2, \beta_0 = 1.8$).

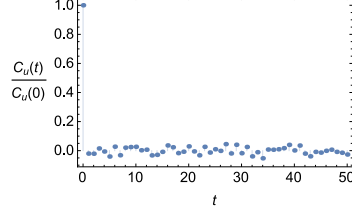


FIG. 9: Correlation function of log-returns u on a daily time scale for AA shares. The time unit of t is days.

IV. CORRELATION FUNCTIONS

For the development of a suitable dynamical model, it is very important to look not only at probability densities but also on correlation functions and memory effects [16]–[21]. In our case there are two types of correlation functions: the one of the original data u_i ,

$$C_u(t) = \frac{1}{N-t} \sum_{i=1}^{N-t} u_i u_{i+t} - \langle u_i \rangle^2 \quad (19)$$

and those of the volatility parameter β_k ,

$$C_\beta(t) = \frac{1}{n-t} \sum_{k=1}^{n-t} \beta_k \beta_{k+t} - \langle \beta_k \rangle^2. \quad (20)$$

Figs. 9-12 show $C_u(t)/C_u(0)$ and $C_\beta(t)/C_\beta(0)$, both for the daily returns as well as for the 1-minute returns. As is illustrated in Fig. 9 and 10, $C_u(t)$ decays almost immediately to zero, both for the daily and minute data.

More interesting is the correlation function $C_\beta(t)$. We did an analysis of the decay rates of correlation functions

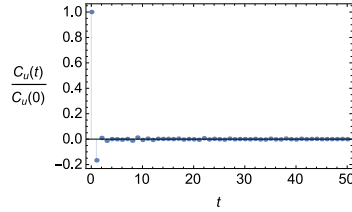


FIG. 10: Correlation function of log-returns u on a time scale of minutes. The time unit of t is minutes.

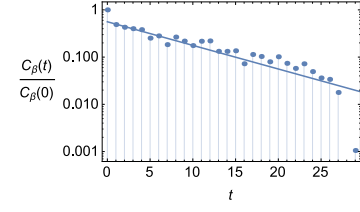


FIG. 11: Correlation function of volatility β for returns on a daily time scale for AA shares.

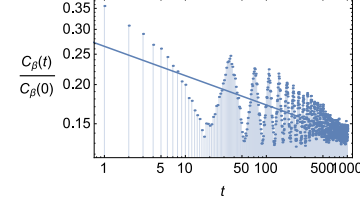


FIG. 12: Correlation function of volatility β for returns on a time scale of minutes.

of the volatility for many different shares from different sectors, the results are summarized in Tab. 1. We observe that the correlation functions of volatility decay in an exponential way for daily returns, $C_\beta(t) \sim e^{-\gamma t}$, whereas for minute return there is a power law decay $C_\beta(t) \sim t^{-\alpha}$ with a periodic modulation, see Figs. 11-12 for the example of AA shares. The strongest correlation decay (largest γ) on the daily scale is observed for shares from basic materials, whereas the power law decay (exponent α) on the small time scale is largest for healthcare shares and shares from the consumer good sector. Note that a strong decay of the volatility correlation function in a sense measures a ‘volatility of a volatility’ and is an interesting quantity to study. The period of oscillations that we observe in figures such as Fig. 12 corresponds (roughly) to one trading day and is consistent with periodic oscillations of intraday volatility reported previously in [19].

Company	Sector	γ	T [Days]	α	T [Mins]
Alcoa Inc. (AA)	basic materials	0.115	18	0.094	11
The Coca-Cola Company (KO)	consumer goods	0.061	15	0.101	13
Bank of America Corporation (BAC)	financial	0.057	20	0.073	13
Johnson & Johnson (JNJ)	healthcare	0.041	16	0.102	13
General Electric Company (GE)	industrial goods	0.068	18	0.062	14
Wal-Mart Stores Inc. (WMT)	services	0.036	16	0.092	13
Intel Corporation (INTC)	technology	0.048	17	0.096	13

Tab. 1: Decay rates of correlation functions of volatility for shares of different sectors

V. SYNTHETIC MODEL

Based on the results of the previous sections, it is desirable to construct a simple superstatistical dynamical model that incorporates the possibility of both lognormal and χ^2 superstatistics on different scales, and allows for different decay patterns of correlation functions.

Here we propose the following model. We start from a linear superstatistical Langevin equation

$$\dot{u} = -\gamma u + \sigma L(t) \quad (21)$$

where $L(t)$ is Gaussian white noise and the ‘inverse temperature’ β , in accordance with Einstein’s theory of Brownian motion, is defined as

$$\beta = \frac{\gamma}{2\sigma^2}. \quad (22)$$

Given a fixed β , the variance $\langle u^2(t) \rangle$ is given by $\frac{1}{2}\langle u^2 \rangle = \beta^{-1}$ for time $t \rightarrow \infty$. However, for the superstatistical version we allow for fluctuations in the parameter β . Then the above Langevin equation is –by construction– superstatistical as we do not keep the parameter β constant but regard it as a random variable that fluctuates on a large time scale. Eq. (21) generates a stochastic process and in the end, after β has taken on many different values, one may rescale the entire time series $u(t)$ to variance 1 using the variance of the complete time series, as we did in eq.(2) and (18)).

Let us now consider $n+1$ Gaussian random variables X_i , $i = 0, 1, 2, \dots, n$ which are statistically independent and have the same variance and mean 0 (except for X_0 which may have potentially a different variance and different mean). We then write β as

$$\beta = \kappa e^{X_0} + (1 - \kappa)(X_1^2 + X_2^2 + \dots + X_n^2), \quad (23)$$

where $\kappa \in [0, 1]$ is a parameter. We now see that if $\kappa = 1$, this system generates lognormal superstatistics, as $\log \beta = X_0$ is a Gaussian random variable. On the other hand, if $\kappa = 0$ this system generates χ^2 -superstatistics with n degrees of freedom, as in this case $\beta = \sum_{i=1}^n X_i^2$ is χ^2 distributed. Choosing any value of $\kappa \in [0, 1]$ one can interpolate between lognormal and χ^2 superstatistics, getting a mixed type of behaviour.

The Gaussian random variables X_i can again be simulated by ordinary linear Langevin equations of the form

$$\dot{X}_i = -\Gamma X_i + \Sigma L_i(t), \quad i = 0, \dots, n \quad (24)$$

For constant Γ and Σ these equations generate the Ornstein Uhlenbeck process, i.e. a Gaussian Markov process with exponential decay of correlation functions. More complicated dynamics, leading e.g. to power law decay of correlation functions, can be constructed if the driving forces in these linear stochastic differential equations are not Gaussian white noise but more complicated correlated processes, or critical maps with a near-vanishing Liapunov exponent [27].

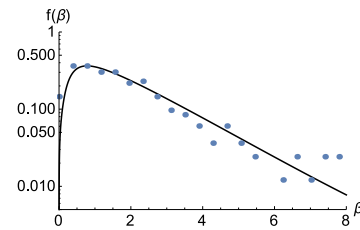


FIG. 13: Mixed distribution fit to $f(\beta)$ with $\kappa = 0.36$ on the daily time scale.

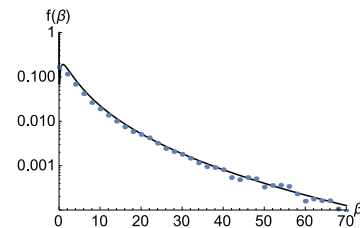


FIG. 14: Mixed distribution fit to $f(\beta)$ with $\kappa = 0.92$ on the time scale of minutes.

Fig. 13 and Fig. 14 show that indeed the observed distributions of $f(\beta)$ for Alcoa shares are best fitted by intermediate distributions (a superposition of a lognormal and χ^2 distribution with appropriate weights). The parameter κ increases if one goes from larger to smaller time scales of returns. The mixed synthetic model is able to reproduce the transition scenario of observed densities from χ^2 superstatistics to lognormal superstatistics in a quantitatively correct way, giving good fits on any time scale.

We did this analysis for a variety of time scales τ of returns, taking again the example of Alcoa shares. In Fig. 15 we show how the parameter κ depends on the time scale of returns. As expected, the parameter κ that best fits the observations decreases as a function of time scale. In fact we observe a logarithmic dependence if the time scale is not too big, see the straight line fit in Fig. 15.

One final remark is at order: One may generalize the superstatistics concept to more complicated local processes that are not locally Gaussian. Indeed, due to correlations present on small time scales, and/or due to a lack of clear time scale separation different distributions than Gaussians may locally be present. In this case one can still superimpose these local distributions by letting a suitable variance parameter fluctuate. It is remarkable, however, that for the financial data analysed here this generalization to more complicated non-Gaussian local processes is not necessary: The simplest superstatistics model based on local Gaussians fits our data well, assuming the interpolating model eq.(23) where the probability distribution of β changes as a function of scale.

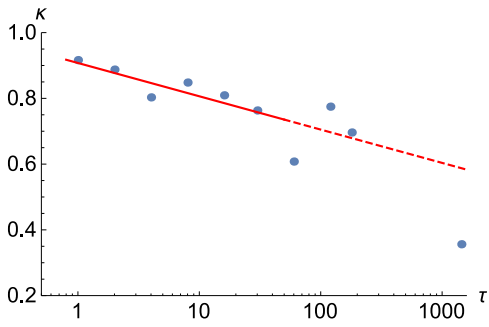


FIG. 15: Parameter κ describing the relative weight of lognormal and χ^2 superstatistics in the mixed model as a function of the time scale τ of return. κ decreases if the time scale τ is increased. For not too big time scales τ a logarithmic dependence is observed: The straight line corresponds to a fit of the first six data points of the form $\kappa = 0.907 - 0.044 \log \tau$.

VI. CONCLUSION

Many investigations of complex systems in the past have focused on the application of a particular statistics, for example q -statistics [22], and then studying the effect of varying system parameters, which may change the entropic index q . Here we have shown that for finan-

cial time series it is sometimes useful to consider broader classes of statistics and even proceed from one class of superstatistics to another when the scale or other system parameters under consideration are changed. The example we considered in detail in this paper were share price returns of various companies. We provided evidence that there is a transition scenario from lognormal superstatistics to χ^2 superstatistics, with lognormal superstatistics giving a better fit to the data on small time scales and χ^2 superstatistics ($= q$ -statistics) on larger time scales. We constructed a hybrid superstatistical model that allows to implement both types of superstatistics, with a weighting parameter κ that describes how far away we are from one of the two cases. Correlation functions of the extracted superstatistical volatility parameter β_k were shown to exhibit different qualitative behavior as a function of the time scale of returns, with exponential decay on large time scales and power law decay on small time scales, modulated by intraday periodicity. The decay parameters of the exponential or power law decay were extracted from the data and were shown to depend slightly on the sector of shares considered. The general transition scenario from lognormal to χ^2 superstatistics as a function of the time scale of returns, however, is a general phenomenon and occurs for all sectors in a similar way.

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